About error bounds in metric spaces

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Abstract The paper presents a general primal space classification scheme of necessary and sufficient criteria for the error bound property incorporating the existing conditions. Several primal space derivative-like objects – slopes – are used to characterize the error bound property of extended-real-valued functions on metric sapces.

1 Introduction

In this paper f is an extended-real-valued function on a metric space X, $f(\bar{x}) = 0$, $S_f := \{x \in X \,|\, f(x) \leq 0\}$, and $f_+(x) := \max(f(x), 0)$. We are looking for characterizations of the *error bound property*.

Definition 1. f has a local error bound at $\bar{x}$ if there exists a $c > 0$ such that

$$d(x, S_f) \leq c f_+(x) \quad \text{for all } x \text{ near } \bar{x}. \tag{1}$$

For the summary of the theory of error bounds and its various applications, the reader is referred to the survey papers [2, 5, 9, 10], as well as the book [1]. Recent extensions to vector-valued functions can be found in [3].

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Property (1) can be equivalently defined in terms of the *error bound modulus* [5]:

$$\mathrm{Er}\,(\bar{x}) := \liminf_{\substack{x \to \bar{x} \\ f(x) > 0}} \frac{f(x)}{d(x, S_f)}, \tag{2}$$

namely, f has a local error bound at $\bar{x}$ if and only if $\mathrm{Er}\, f(\bar{x}) > 0$. Constant (2) provides a quantitative characterization of this property.

2 Slopes

Primal space characterizations of error bounds can be formulated in terms of slopes.

Recall that the (strong) *slope* [4] of f at x ($|f(x)| < \infty$) is defined as

$$|\nabla f|(x) := \limsup_{u \to x} \frac{(f(x) - f(u))_+}{d(u,x)}. \tag{3}$$

The following modifications of (3) can be convenient for characterizing the error bound property:

$$|\nabla f|^0(\bar{x}) = \liminf_{x \to \bar{x}} \frac{f(x) - f(\bar{x})}{\|x - \bar{x}\|}, \tag{4}$$

$$\overline{|\nabla f|}(\bar{x}) = \liminf_{x \to \bar{x},\, f(x) \to f(\bar{x})} |\nabla f|(x), \tag{5}$$

$$\overline{|\nabla f|}^{>}(\bar{x}) = \liminf_{x \to \bar{x},\, f(x) \downarrow f(\bar{x})} |\nabla f|(x), \tag{6}$$

$$\overline{|\nabla f|}^{\diamond}(\bar{x}) := \liminf_{x \to \bar{x},\, f(x) \downarrow f(\bar{x})} \sup_{u \neq x} \frac{(f(x) - f_+(u))_+}{d(u,x)}. \tag{7}$$

Constants (4)–(7) are called the *internal slope*, *strict slope*, *strict outer slope*, and *uniform strict slope* of f at $\bar{x}$ respectively.

The relationships between the constants are straightforward.

Proposition 1. *(i)* $\overline{|\nabla f|}(\bar{x}) \le \overline{|\nabla f|}^{>}(\bar{x}) \le \overline{|\nabla f|}^{\diamond}(\bar{x})$.
(ii) $|\nabla f|(\bar{x}) = (-|\nabla f|^0(\bar{x}))_+$.
(iii) $|\nabla f|^0(\bar{x}) \le \overline{|\nabla f|}^{\diamond}(\bar{x})$.
(iv) If $|\nabla f|^0(\bar{x}) > 0$ *then* $|\nabla f|^0(\bar{x}) = \mathrm{Er}\, f(\bar{x})$.

The inequalities in Proposition 1 can be strict.

Example 1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} 0 \text{ if } x < 0, \\ x \text{ if } x \ge 0. \end{cases}$$

Obviously $\overline{|\nabla f|}(0) = |\nabla f|(0) = |\nabla f|^0(0) = 0$. At the same time, $\overline{|\nabla f|}^{>}(0) = \overline{|\nabla f|}^{\diamond}(0) = 1$.

The next example is a modification of the corresponding one in [6].

Example 2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} -x \text{ if } x \le 0, \\ \dfrac{1}{i} \quad \text{if } \dfrac{1}{i+1} < x \le \dfrac{1}{i},\ i = 1,2,\ldots, \\ x \quad \text{if } x > 1. \end{cases}$$

Obviously $\overline{|\nabla f|}^{>}(0) = |\nabla f|(0) = 0$. At the same time, $\overline{|\nabla f|}^{\diamond}(0) = |\nabla f|^{0}(0) = 1$.

The function in the above example is discontinuous. However, the second inequality in Proposition 1 (i) can be strict for continuous and even Lipschitz continuous functions. The function in the next example is piecewise linear and Clarke regular at 0 (that is, directionally differentiable, and its Clarke generalized directional derivative coincides with the usual one).

Example 3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} -x & \text{if } x \le 0, \\ x\left(1+\dfrac{1}{i}\right) - \dfrac{1}{i(i+1)} & \text{if } \dfrac{1}{i+1} < x \le \dfrac{1}{i+1} + \dfrac{1}{(i+1)^2},\ i = 1,2,\ldots, \\ \dfrac{1}{i} & \text{if } \dfrac{1}{i+1} + \dfrac{1}{(i+1)^2} < x \le \dfrac{1}{i},\ i = 1,2,\ldots, \\ x & \text{if } x > 1. \end{cases}$$

f is everywhere Fréchet differentiable except for a countable number of points. One can find a point $x > 0$ arbitrarily close to 0 with $|\nabla f|(x) = 0$ (on a horizontal part of the graph). The slopes of non-horizontal parts of the graph decrease monotonously to 1 as $x \downarrow 0$. It is not difficult to check that $\overline{|\nabla f|}^{>}(0) = |\nabla f|(0) = 0$ while $\overline{|\nabla f|}^{\diamond}(0) = |\nabla f|^{0}(0) = 1$.

If f is convex then the second inequality in Proposition 1 (i) holds as equality.

For the function f in Example 1, it holds $|\nabla f|(0) < \overline{|\nabla f|}^{>}(0)$. In the nonconvex case one can also have the opposite inequality.

Example 4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows:

$$f(x) = \begin{cases} x & \text{if } x < 0, \\ x^2 & \text{if } x \ge 0. \end{cases}$$

Obviously $|\nabla f|(0) = 1$ while $\overline{|\nabla f|}^{>}(0) = 0$. Note that despite slope $|\nabla f|(0)$ being positive, the function in this example does not have a local error bound at 0. Hence, condition $|\nabla f|(\bar{x}) > 0$ is not in general sufficient for the error bound property to hold at $\bar{x}$.

3 Error bound criteria

The next theorem generalizes and strengthens [5, Theorem 2].

Theorem 1. *(i)* $\operatorname{Er} f(\bar{x}) \le \overline{|\nabla f|}^{\diamond}(\bar{x})$.
(ii) If X is complete and f_+ is lower semicontinuous near $\bar{x}$, then $\operatorname{Er} f(\bar{x}) = \overline{|\nabla f|}^{\diamond}(\bar{x})$.

Proof. (i) If $\operatorname{Er} f(\bar{x}) = 0$ or $\overline{|\nabla f|}^{\diamond}(\bar{x}) = \infty$, the conclusion is trivial. Let $0 < \gamma < \operatorname{Er} f(\bar{x})$ and $\overline{|\nabla f|}^{\diamond}(\bar{x}) < \infty$. We are going to show that $\overline{|\nabla f|}^{\diamond}(\bar{x}) \ge \gamma$. By (2), there is a $\delta > 0$ such that

$$\frac{f(x)}{d(x,S_f)} > \gamma. \tag{8}$$

for any $x \in B_\delta(\bar{x})$ with $f(x) > f(\bar{x})$. Take any $x \in B_\delta(\bar{x})$ with $f(\bar{x}) < f(x) \le f(\bar{x}) + \delta$ (Such points x exist since $\overline{|\nabla f|}^{\diamond}(\bar{x}) < \infty$.) By (8), one can find a $w \in S_f$ such that

$$\frac{f(x)}{d(x,w)} > \gamma.$$

It follows that $\overline{|\nabla f|}^{\diamond}(\bar{x}) \ge \gamma$.

(ii) Let X be complete and f_+ be lower semicontinuous near $\bar{x}$. Thanks to (i), we only need to prove that $\operatorname{Er} f(\bar{x}) \ge \overline{|\nabla f|}^{\diamond}(\bar{x})$. If $\operatorname{Er} f(\bar{x}) = \infty$, the inequality is trivial. Let $\operatorname{Er} f(\bar{x}) < \gamma < \infty$. Chose a $\delta > 0$ such that f_+ is lower semicontinuous on $B_{(\gamma^{-1}+1)\delta}(\bar{x})$. Then by (2), there is an $x \in B_{\delta \min(1/2,\gamma^{-1})}(\bar{x})$ such that

$$0 < f(x) < \gamma d(x,S_f).$$

Put $\varepsilon = f(x)$. Then $f_+(x) \le \inf f_+ + \varepsilon$. Applying to f_+ the Ekeland variational principle with an arbitrary $\lambda \in (\gamma^{-1}\varepsilon, d(x,S_f))$, one can find a w such that $f(w) \le f(x)$, $d(w,x) \le \lambda$ and

$$f_+(u) + (\varepsilon/\lambda) d(u,w) \ge f_+(w), \qquad \forall u \in B_{(\gamma^{-1}+1)\delta}(\bar{x}). \tag{9}$$

Obviously,

$$\begin{gathered} d(w,x) < d(x,S_f) \le d(x,\bar{x}), \\ d(w,\bar{x}) \le d(w,x) + d(x,\bar{x}) < 2d(x,\bar{x}) \le \delta, \\ f(w) \le f(x) < \gamma d(x,\bar{x}) \le \delta. \end{gathered} \tag{10}$$

Besides, $f(w) > 0$ due to the first inequality in (10). It follows from (9) that

$$f(w) \le f_+(u) + (\varepsilon/\lambda) d(u,w) \le \gamma d(u,w)$$

for all $u \in B_{(\gamma^{-1}+1)\delta}(\bar{x})$. If $u \notin B_{(\gamma^{-1}+1)\delta}(\bar{x})$, then $d(u,w) > (\gamma^{-1}+1)\delta - d(w,\bar{x}) > \gamma^{-1}\delta$, and consequently

$$f(w) < \delta < \gamma d(u,w)$$

Thus, in both cases

$$\sup_{u\neq w}\frac{f(w)-f_+(u)}{d(u,w)}\le\gamma.$$

This implies the inequality $\overline{|\nabla f|}^{\diamond}(\bar{x})\le \operatorname{Er} f(\bar{x})$. $\square$

Without lower semicontinuity, the inequality in Theorem 1 (i) can be strict.

Example 5. Let $f:\mathbb{R}\to\mathbb{R}$ be defined as follows:

$$f(x)=\begin{cases}-3x & \text{if } x\le 0,\\ 3x-\dfrac{1}{2^i} & \text{if } \dfrac{1}{2^{i+1}}<x\le\dfrac{1}{2^i},\ i=0,1,\dots,\\ 2x & \text{if } x>1.\end{cases}$$

Obviously, $\operatorname{Er} f(0)=1$ while $\overline{|\nabla f|}^{\diamond}(0)=3$.

Example 6. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined as follows:

$$f(x_1,x_2)=\begin{cases}x_1+x_2 & \text{if } x_1>0, x_2>0,\\ -x_1 & \text{if } x_1>0, x_2\le 0,\\ -x_2 & \text{if } x_2>0, x_1\le 0,\\ 0 & \text{otherwise,}\end{cases}$$

and let $\mathbb{R}^2$ be equipped with the Euclidean norm. The function is discontinuous on the set $\{(t,0)\in\mathbb{R}^2: t>0\}\cup\{(0,t)\in\mathbb{R}^2: t>0\}$. Then $\operatorname{Er} f(0)=2$ and $\overline{|\nabla f|}^{\diamond}(0)=3$.

In view of Theorem 1, inequality $\overline{|\nabla f|}^{\diamond}(\bar{x})>0$ provides a necessary and sufficient error bound criterion for lower semicontinuous functions on complete metric spaces. In a slightly different form, a similar condition for the calmness [q] property of level set maps first appeared in [8, Proposition 3.4]; see also [7, Corollary 4.3].

Taking into account Proposition 1, inequalities

$$|\nabla f|^0(\bar{x})>0,\quad \overline{|\nabla f|}(\bar{x})>0\quad\text{and}\quad \overline{|\nabla f|}^{>}(\bar{x})>0$$

provide sufficient error bound criteria

The relationships among the primal space error bound criteria are illustrated in Fig. 1 (X is complete and f_+ is lower semicontinuous near $\bar{x}$).

In Banach spaces, it is possible to formulate corresponding dual space error bound criteria in terms of subdifferential slopes [5].

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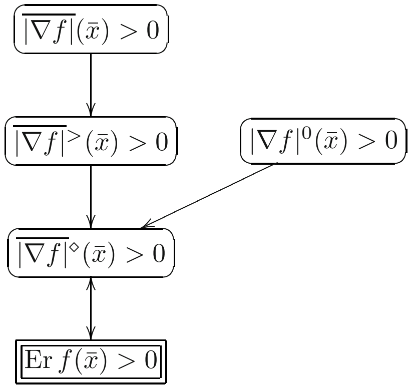

Fig. 1 Primal space criteria

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